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A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems

LEONARD MEIROVITCH*

Virginia Polytechnic Institute and State University, Blacksburg, Va.

This paper is concerned with the motion characteristics of gyroscopic systems described by $2n$ first-order ordinary differential equations defined by two real nonsingular matrices, one symmetric and one skew symmetric. The equations describe the motion of a spinning body containing elastic parts. Taking advantage of the special nature of the problem, a new method of solution of the associated eigenvalue problem is developed, whereby the eigenvalue problem is transformed into one in terms of two real symmetric matrices. A basic new concept is introduced in the form of a state vector that includes the rotational motion of the structure as a whole and the elastic motion relative to the rotating frame. Orthogonality relations and an expansion theorem are developed in terms of such eigenvectors. As an illustration of the method, the natural frequencies and natural modes of a spinning spacecraft containing elastic parts are calculated.

Introduction

THE theory for the eigenvalue problem for nonspinning systems with elastic restoring forces is well developed. By comparison, the eigenvalue problem for spinning systems containing elastic parts has received very little attention. Such systems can be defined in terms of two matrices, one symmetric and one skew symmetric, and belong to the general class of *gyroscopic systems.* With the advent of spinning spacecraft, the eigenvalue problem associated with gyroscopic systems has gained increasing importance, although work on the subject remains scant.

The problem of gyroscopic systems is mentioned by Frazer, Duncan, and Collar[1] in conjunction with Lagrange's equations for systems referred to rotating axes, but the discussion does not go beyond the formulation of the problem. Lancaster[2] points out that the eigenvalues of an undamped gyroscopic system are pure imaginary complex conjugates, and that the associated eigenvectors are also complex conjugates, but presents no special algorithm for the solution of the eigenvalue problem other than the general ones. The problem of a damped gyroscopic system, simulating a spinning satellite containing elastic parts, was discussed by Meirovitch and Nelson.[3] Reference 3 used a Rayleigh-Ritz approach to discretize continuous elastic members. The eigenvalues of the complete rotational system were obtained by solving the characteristic equation numerically, but no attempt was made to develop general methods of solution. The equations of motion for a complex gyroscopic system were derived by Likins.[4] Reference 4 proposes a solution in terms of the eigenvectors for the elastic displacements alone, with the terms due to the rotational motion of the spacecraft being regarded as external excitations. Consistent with this approach, Ref. 4 provides a brief survey of possible ways of solving the eigenvalue problem associated with nonrotating damped structures, but no numerical example illustrating the approach is presented. Using the approach of Ref. 4, Patel and Seltzer[5] propose a computer program to solve the eigenvalue problem associated with the elastic displacements alone. The case of constant angular velocity receives special attention. Following the same line of thought as in Refs. 4 and 5, Gupta[6,7] also concerns himself with an eigenvalue problem for elastic deformations alone. References 6 and 7 avoid the question of rotational degrees of freedom entirely by considering mathematical models consisting of elastic structures spinning with constant angular velocity about an axis fixed in space. These are computationally oriented papers in which the author proposes an algorithm that takes advantage of the fact that the eigenvalue problem is defined by highly banded matrices, but not explicitly of the fact that one matrix is symmetric and the other is skew symmetric. Although the mathematical models are intended to represent spinning space structures, the fact that the

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* Professor, Department of Engineering Science and Mechanics. Associate Fellow AIAA.

rotational degrees of freedom are artificially suppressed renders such mathematical models highly unrealistic. Yet, if the structures considered were permitted to nutate instead of being constrained to rotate uniformly about a fixed axis, the matrices defining the eigenvalue problem would no longer be banded and the proposed algorithm would no longer be applicable. It should be stressed that none of Refs. 4–7 answers the real need, namely, that of producing the natural frequencies and natural modes of the complete rotating structure. An approach that does not include the rotational motion in the eigenvalue problem cannot provide the basis for a genuine modal analysis for space structures, as the associated eigenvectors expansion does not uncouple the differential equations of motion.

The present paper is concerned with a flexible gyroscopic system described by a set of $2n$ first-order ordinary differential equations defined by one symmetric and one skew symmetric matrix. The motion of a body containing elastic parts and spinning freely in space, such as that of Ref. 3, is described by this type of equation. A new approach to the solution of the associated eigenvalue problem is developed. To this end, the concept of a state vector that includes the rotational motion of the structure as a whole and the elastic motion relative to the rotating frame is introduced. Then advantage is taken of the fact that the system is defined by one symmetric and one skew symmetric matrix to reduce the eigenvalue problem to a standard form, namely, one in terms of two real symmetric matrices for both the real and the imaginary part of the eigenvectors. This latter type of eigenvalue problem resembles in structure that of a nonrotating structure and can be solved by a large variety of existing methods. A proof of the general orthogonality of the eigenvectors is provided and an expansion theorem particularly suited for gyroscopic systems is developed, the latter having important implications in modal analysis. Indeed, based on the theory of this paper, a true modal analysis for the response of structures has been developed.[8]

The theory developed in this paper should go a long way toward enhancing not only the mathematical analysis but also the understanding of the behavior of flexible gyroscopic systems. In addition to problems of dynamics of flexible spacecraft, the method should find wide application in missile dynamics.

As an illustration, the eigenvalue problem corresponding to the gyroscopic system of Ref. 3 is solved. This represents the first time that the natural frequencies and natural modes of a complete system have been obtained in a systematic manner.

The Eigenvalue Problem and Its Reduction to Standard Form

Let us consider a dynamical system described by $2n$ first-order homogeneous differential equations having the matrix form

$$I\dot{\mathbf{x}}(t)+G\mathbf{x}(t)=\mathbf{0} \tag{1}$$

where I and G are $2n\times 2n$ real nonsingular matrices, the first symmetric and the second skew symmetric, $\mathbf{x}(t)$ is a $2n$-dimensional state vector, and $\mathbf{0}$ is the $2n$-dimensional null vector. We shall seek a solution of Eq. (1) in the form

$$\mathbf{x}(t)=\mathbf{x}\,e^{\lambda t} \tag{2}$$

where $\mathbf{x}$ is a $2n$-dimensional constant vector with complex elements and λ is a constant complex number. Introducing Eq. (2) into Eq. (1) and dividing through by $e^{\lambda t}$, we obtain

$$\lambda I\mathbf{x}+G\mathbf{x}=\mathbf{0} \tag{3}$$

which represents an eigenvalue problem of order $2n$.

The solution of the eigenvalue problem (3) consists of $2n$ eigenvalues λ_r and associated eigenvectors $\mathbf{x}_r$ ($r=1,2,\ldots,2n$). Because of the fact that I is symmetric and G is skew symmetric, the solution of the eigenvalue problem possesses certain properties that are useful in obtaining numerical solutions. To establish these properties, let us consider the characteristic equation

$$\det(\lambda I+G)=0 \tag{4}$$

Because the determinant of a matrix is equal to the determinant of the transposed matrix, we must also have

$$\det(\lambda I+G)^T=\det(\lambda I^T+G^T)=\det(\lambda I-G)=0 \tag{5}$$

so that, *if λ is an eigenvalue, then $-\lambda$ is also an eigenvalue.*

Next we wish to show that the eigenvalues are pure imaginary, hence they occur in pairs of complex conjugates, and that the associated eigenvectors are also complex conjugates. Divergent solutions are excluded from this discussion. To this end, let us consider a given solution, λ_r, $\mathbf{x}_r$ of the eigenvalue problem (3), where the solution satisfies the equation

$$\lambda_r I\mathbf{x}_r+G\mathbf{x}_r=\mathbf{0} \tag{6}$$

and multiply Eq. (6) on the left by the complex conjugate $\bar{\mathbf{x}}_r^T$ to obtain

$$\lambda_r\bar{\mathbf{x}}_r^T I\mathbf{x}_r+\bar{\mathbf{x}}_r^T G\mathbf{x}_r=0 \tag{7}$$

Next, let us introduce the notation

$$\mathbf{x}_r=\mathbf{y}_r+i\mathbf{z}_r,\ \bar{\mathbf{x}}_r=\mathbf{y}_r-i\mathbf{z}_r \tag{8}$$

where the real vectors $\mathbf{y}_r$ and $\mathbf{z}_r$ represent the real part and the imaginary part of $\mathbf{x}_r$, respectively. Introducing Eqs. (8) into Eq. (7), it is easy to verify that

$$\lambda_r(\mathbf{y}_r^T I\mathbf{y}_r+\mathbf{z}_r^T I\mathbf{z}_r)+2i\mathbf{y}_r^T G\mathbf{z}_r=0 \tag{9}$$

But the triple matrix products $\mathbf{y}_r^T I\mathbf{y}_r+\mathbf{z}_r^T I\mathbf{z}_r$ and $\mathbf{y}_r^T G\mathbf{z}_r$ represent real numbers. It follows that λ_r is pure imaginary. Because $-\lambda_r$ is also an eigenvalue, *the system eigenvalues consist of n pairs of pure imaginary complex conjugates.* We shall denote the eigenvalues by $\pm i\omega_r$ ($r=1,2,\ldots,n$), where ω_r are recognized as the natural frequencies of the system. It is easy to see that if $i\omega_r$ and $\mathbf{x}_r$ represent a solution of the eigenvalue problem, then $-i\omega_r$ and $\bar{\mathbf{x}}_r$ also constitute a solution. It follows that *the system eigenvectors consist of n pairs of complex conjugates $\mathbf{x}_r$ and $\bar{\mathbf{x}}_r$,* corresponding to the eigenvalues $i\omega_r$ and $-i\omega_r$, respectively.

In developing an algorithm for the solution of the eigenvalue problem (3), it is desirable to work with real quantities instead of complex quantities. The following developments are directed to this goal.

Let us consider a given solution $\lambda_r=i\omega_r$, $\mathbf{x}_r$ of the eigenvalue problem (3), so that

$$i\omega_r I\mathbf{x}_r+G\mathbf{x}_r=\mathbf{0} \tag{10}$$

Inserting $\mathbf{x}_r=\mathbf{y}_r+i\mathbf{z}_r$ into Eq. (10), we have

$$i\omega_r I(\mathbf{y}_r+i\mathbf{z}_r)+G(\mathbf{y}_r+i\mathbf{z}_r)=\mathbf{0} \tag{11}$$

so that, separating the real and imaginary parts of Eq. (11), we obtain two equations in terms of real quantities

$$-\omega_r I\mathbf{z}_r+G\mathbf{y}_r=\mathbf{0} \tag{12a}$$

$$\omega_r I\mathbf{y}_r+G\mathbf{z}_r=\mathbf{0} \tag{12b}$$

From Eq. (12a), we can write

$$\mathbf{z}_r=(1/\omega_r)I^{-1}G\mathbf{y}_r \tag{13}$$

so that, inserting Eq. (13) into Eq. (12b), we obtain

$$\omega_r^2 I\mathbf{y}_r=K\mathbf{y}_r,\qquad r=1,2,\ldots,n \tag{14}$$

where

$$K=-GI^{-1}G=G^TI^{-1}G \tag{15}$$

is a real symmetric matrix, $K=K^T$, because I and G are real, the first being symmetric and the second skew symmetric. Similarly, Eq. (12b) yields

$$\mathbf{y}_r=-(1/\omega_r)I^{-1}G\mathbf{z}_r \tag{16}$$

so that, inserting Eq. (16) into Eq. (12a), we have

$$\omega_r^2 I\mathbf{z}_r=K\mathbf{z}_r,\quad r=1,2,\ldots,n \tag{17}$$

From Eqs. (14) and (17), we conclude that the eigenvalue problem defined by I and K yields both the real part $\mathbf{y}_r$ and the imaginary part $\mathbf{z}_r$ of the eigenvector $\mathbf{x}_r$ ($r=1,2,\ldots,n$). Hence, *the eigenvalue problem (3) defined by one real symmetric and one real skew symmetric matrix and possessing complex solutions has been reduced to the eigenvalue problem (14) and (17) defined by two real symmetric matrices and possessing real solutions.* Because the problem (14) and (17) is of order $2n$ it must possess $2n$

solutions. They consist of n pairs of repeated eigenvalues ω_r^2 and n pairs of associated eigenvectors $\mathbf{y}_r$ and $\mathbf{z}_r$ $(r = 1, 2, \ldots, n)$.

We shall refer to the eigenvalue problem defined by the symmetric matrices I and K as being in a *standard form*, because this form is similar in structure to the eigenvalue problem for nonrotating systems. Of course, for nonrotating systems the order of the matrices I and K is only n, where I is simply the inertia matrix and K the stiffness matrix and $\mathbf{y}_r$ is only an n-dimensional configuration vector instead of a $2n$-dimensional state vector. Moreover, for nonrotating systems there is no imaginary part $\mathbf{z}_r$.

The power of reducing the eigenvalue problem (3) to a standard form, Eqs. (14) and (17), lies in the fact that it eliminates the need to solve an eigenvalue problem known to possess complex solutions, and for which there are few satisfactory computational algorithms available, in favor of solving an eigenvalue problem known to possess real solutions, and for which there is available a large variety of satisfactory algorithms. The use of computer algorithms using real algebra, as opposed to complex algebra, results in substantial savings in computational time. Several such algorithms, including the power method and the Jacobi method, can be found in the texts by Meirovitch.[9,10] Caution must be exercised in using these methods, however, because of the repeated roots. If the algorithm yields two eigenvectors for every root ω_r^2, then they can be regarded as being $\mathbf{y}_r$ and $\mathbf{z}_r$. This is the case with the Jacobi method, a method based on matrix diagonalization that iterates to all the eigenvalues and eigenvectors simultaneously provided all the eigenvectors are independent, including those corresponding to the repeated roots. On the other hand, if the method yields only one eigenvector at a time, as in the case of the power method using matrix deflation, then the eigenvector can be regarded as being $\mathbf{y}_r$ and Eq. (13) can be used to obtain $\mathbf{z}_r$, thus eliminating the need for an iteration sequence associated with $\mathbf{z}_r$.

Many texts, particularly those on numerical analysis, use a more simple standard form, namely, one defined by only one symmetric matrix instead of two. Clearly, this represents a special case of the eigenvalue problem (14) and (17) in the sense that the inertia matrix I is proportional to the identity matrix of order $2n$. Under certain circumstances, Eqs. (14) and (17) can be reduced to the more simple standard form. We shall assume that this is possible for the systems that interest us and verify this assumption later. Hence, let us write Eq. (14) in the form

$$\omega_r^2 I^{1/2} I^{1/2} \mathbf{y}_r = K I^{-1/2} I^{1/2} \mathbf{y}_r \tag{18}$$

introduce the linear transformation

$$\mathbf{y}_r' = I^{1/2} \mathbf{y}_r \tag{19}$$

multiply Eq. (18) on the left by $I^{-1/2}$, and obtain

$$\omega_r^2 \mathbf{y}_r' = K' \mathbf{y}_r', \quad r = 1, 2, \ldots, n \tag{20}$$

where

$$K' = I^{-1/2} K I^{-1/2} \tag{21}$$

is a real symmetric matrix because I and K are real symmetric matrices. In a similar way, we have

$$\omega_r^2 \mathbf{z}_r' = K' \mathbf{z}_r', \qquad r = 1, 2, \ldots, n \tag{22}$$

where

$$\mathbf{z}_r' = I^{1/2} \mathbf{z}_r \tag{23}$$

Clearly, the transformation from Eqs. (14) and (17) to Eqs. (20) and (22) is possible only if $I^{1/2}$ and $I^{-1/2}$ exist. We shall show in the next section the necessary conditions for $I^{1/2}$ and $I^{-1/2}$ to exist. The eigenvalue problem (20) and (22) represents a more simple *standard form* than that described by Eqs. (14) and (17). Whereas Eqs. (20) and (22) yield the actual eigenvalues, to obtain the actual eigenvectors, we must write

$$\mathbf{y}_r = I^{-1/2} \mathbf{y}_r', \quad \mathbf{z}_r = I^{-1/2} \mathbf{z}_r', \quad r = 1, 2, \ldots, n \tag{24}$$

A large variety of algorithms for the solution of the eigenvalue problem (20) and (22) can be found in many texts on numerical analysis, such as that by Ralston[11] and that by Wilkinson.[12] Useful algorithms include the power method, Jacobi's method, Given's method, Householder's method, the QR algorithm, inverse iteration, etc. The author of this paper is presently assessing the relative merits of many of these methods for the solution of the eigenvalue problem associated with rotating structures possessing large degrees of freedom.

The eigenvectors $\mathbf{y}_r$ and $\mathbf{z}_r$ possess certain properties that are not only interesting but also essential to various computational methods. The most important one is the orthogonality property, with the implied independence of the eigenvectors. Some of these properties are demonstrated in the next section.

Properties of Eigenvalue Problem Solution

The theory associated with the eigenvalue problem of a real symmetric matrix is well developed and properties of its solution are discussed amply in standard texts on linear algebra such as that by Murdoch.[13] In proving certain properties of our gyroscopic systems, we shall rely on various theorems given in Ref. 13.

We recall from the preceding section that K' is a real symmetric matrix, but the existence of K' hinges on the existence of $I^{1/2}$ and $I^{-1/2}$. Next we wish to verify whether $I^{1/2}$ and $I^{-1/2}$ do indeed exist. The matrix $I^{1/2}$ can be interpreted as the square root of I. Let us denote it by E

$$I^{1/2} = E \tag{25}$$

and assume that a nonsingular transformation matrix P exists such that

$$P^{-1} I P = R \tag{26}$$

where R is a diagonal matrix of order $2n$. Then I is said to be diagonalizable and the matrices I and R are said be similar. Because I is a real symmetric matrix, it is indeed diagonalizable (see Theorem 6.12 of Ref. 13). Moreover, because I is similar to the diagonal matrix R, the diagonal elements of R are the eigenvalues of I (see Corollary to Theorem 6.4 of Ref. 13). However, Eq. (25) implies that $I = E^2$, so that Eq. (26) can be written as

$$P^{-1} E^2 P = (P^{-1} E P)(P^{-1} E P) = R \tag{27}$$

from which we conclude that

$$P^{-1} I^{1/2} P = R^{1/2} \tag{28}$$

Hence, $I^{1/2}$ exists if $R^{1/2}$ exists, where $R^{1/2}$ is a diagonal matrix with its diagonal elements equal to the square root of the homologous elements of R. It follows that for $I^{1/2}$ to exist it is necessary that all the eigenvalues of I be positive. Note that complex elements are excluded in $R^{1/2}$. But all the eigenvalues of I are positive if and only if I is a positive definite matrix (see Theorem 6.15 of Ref. 13). Hence, let us assume that I is indeed positive definite, so that $I^{1/2}$ exists. Then it is easy to verify that

$$P^{-1} I^{-1/2} P = R^{-1/2} \tag{29}$$

where $R^{-1/2}$ is simply the inverse of $R^{1/2}$. Hence, if I is positive definite, the matrix $I^{-1/2}$ also exists. This leads us to the conclusion that *if I is positive definite, the matrix K', Eq. (21), exists.*

Next we shall show that if I is positive definite K' not only exists but it is also positive definite. To this end, let us use Eqs. (15) and (21) and write K' in the form

$$K' = I^{-1/2} G^T I^{-1} G I^{-1/2} = (I^{-1/2} G^T I^{-1/2})(I^{-1/2} G I^{-1/2}) \tag{30}$$

Hence, introducing the notation

$$Q = I^{-1/2} G I^{-1/2} \tag{31}$$

where Q is a nonsingular matrix because $I^{-1/2}$ and G are nonsingular, Eq. (30) can be written as the matrix product

$$K' = Q^T Q \tag{32}$$

from which it follows that K' *is a positive definite matrix* (see Theorem 6.16 of Ref. 13), provided that I is positive definite. Invoking once again Theorem 6.15 of Ref. 13, we conclude that *if I is positive definite, then all the eigenvalues of K' are positive, so that no ω_r can be zero.* Hence, our earlier exclusion of divergent motion implied the assumptions that I is positive definite and that G is nonsingular.

The fact that K' is a real symmetric matrix has important implications as far as the properties of the eigenvectors $\mathbf{y}_r'$ and $\mathbf{z}_r'$ are concerned. Indeed, because K' is a real symmetric matrix, two eigenvectors corresponding to two distinct eigenvalues are orthogonal in an ordinary sense (see Theorem 6.13 of Ref. 13), or

$$\mathbf{y}_r'^T\mathbf{y}_s' = 0, \quad \mathbf{y}_r'^T\mathbf{z}_s' = 0, \quad \mathbf{z}_r'^T\mathbf{y}_s' = 0, \quad \mathbf{z}_r'^T\mathbf{z}_s' = 0 \quad \text{for } \omega_r^2 \neq \omega_s^2 \tag{33a}$$

Actually, the second of the preceding orthogonality relations is valid also for repeated roots, i.e., if a repeated eigenvalue of a real symmetric matrix is of multiplicity p, then there are exactly p mutually orthogonal eigenvectors corresponding to that eigenvalue (see Theorem 6.14 of Ref. 13). Because in our case every ω_r^2 is a double root with corresponding eigenvectors $\mathbf{y}_r'$ and $\mathbf{z}_r'$, it follows that the second and third of Eqs. (33a) can be liberalized so as to read

$$\mathbf{y}_r'^T\mathbf{z}_s' = 0, \quad \mathbf{z}_r'^T\mathbf{y}_s' = 0 \tag{33b}$$

Recalling transformations (19) and (23), however, we can write

$$\mathbf{y}_r^T I\mathbf{y}_s = 0, \quad \mathbf{z}_r^T I\mathbf{z}_s = 0 \quad \text{for} \quad \omega_r^2 \neq \omega_s^2 \tag{34a}$$

and

$$\mathbf{y}_r^T I\mathbf{z}_s = 0, \quad \mathbf{z}_r^T I\mathbf{y}_s = 0 \tag{34b}$$

or, *the real and imaginary parts of the eigenvectors* $\mathbf{x}_r$ ($r = 1, 2, \ldots, n$) *form a set of* $2n$ *eigenvectors orthogonal with respect to the matrix* I. The implication of this statement is that *the* $2n$ *eigenvectors* $\mathbf{y}_r$ *and* $\mathbf{z}_r$ ($r = 1, 2, \ldots, n$) *are independent.* It should be pointed out that, because $\mathbf{y}_r$ and $\mathbf{z}_r$ are two eigenvectors corresponding to the same eigenvalue ω_r^2, *any linear combination of* $\mathbf{y}_r$ *and* $\mathbf{z}_r$ *is also an eigenvector.* Clearly, only two eigenvectors corresponding to a given ω_r^2 ($r = 1, 2, \ldots, n$) are independent.

It will prove convenient to normalize the eigenvectors $\mathbf{y}_r$ and $\mathbf{z}_r$ so as to satisfy the equations

$$\mathbf{y}_r^T I\mathbf{y}_r = \mathbf{z}_r^T I\mathbf{z}_r = 1, \quad r = 1, 2, \ldots, n \tag{35}$$

in which case $\mathbf{y}_r$ and $\mathbf{z}_r$ are orthonormal, so that Eqs. (34a) and (35) can be combined into

$$\mathbf{y}_r^T I\mathbf{y}_s = \mathbf{z}_r^T I\mathbf{z}_s = \delta_{rs}, \quad r, s = 1, 2, \ldots, n \tag{36}$$

where δ_{rs} is the Kronecker delta.

Expansion Theorem

Because the eigenvectors $\mathbf{y}_r$ and $\mathbf{z}_r$ ($r = 1, 2, \ldots, n$) are orthogonal (with respect to the matrix I), they constitute a set of $2n$ independent vectors, hence they form a basis in a $2n$-dimensional vector space. Accordingly, any $2n$-dimensional vector $\mathbf{v}$ in the space can be expanded in terms of the eigenvectors $\mathbf{y}_r$ and $\mathbf{z}_r$ ($r = 1, 2, \ldots, n$) in the form

$$\mathbf{v} = \sum_{r=1}^{n} a_r\mathbf{y}_r + \sum_{r=1}^{n} b_r\mathbf{z}_r \tag{37}$$

where a_r and b_r are real coefficients. To obtain the value of the coefficients a_r for a given vector $\mathbf{v}$, let us multiply Eq. (37) on the left by $\mathbf{y}_s^T I$, and obtain

$$\mathbf{y}_s^T I\mathbf{v} = \sum_{r=1}^{n} a_r\mathbf{y}_s^T I\mathbf{y}_r + \sum_{r=1}^{n} b_r\mathbf{y}_s^T I\mathbf{z}_r \tag{38}$$

Assuming that the eigenvectors are normalized according to Eq. (35), and invoking the orthogonality conditions, Eqs. (34b) and (36), we simply have

$$a_r = \mathbf{y}_r^T I\mathbf{v}, \quad r = 1, 2, \ldots, n \tag{39a}$$

In a similar fashion, it is easy to show that

$$b_r = \mathbf{z}_r^T I\mathbf{v}, \quad r = 1, 2, \ldots, n \tag{39b}$$

Hence, we can state the following *expansion theorem:* Any arbitrary $2n$-dimensional vector $\mathbf{v}$ can be expressed as a linear combination of the eigenvectors $\mathbf{y}_r$ and $\mathbf{z}_r$ ($r = 1, 2, \ldots, n$) of the form (37), where the coefficients a_r and b_r ($r = 1, 2, \ldots, n$) are given by Eqs. (39).

The expansion theorem, Eqs. (37) and (39), together with the orthogonality relations, Eqs. (34b) and (36), form the basis of the modal analysis for gyroscopic systems developed in Ref. 8.

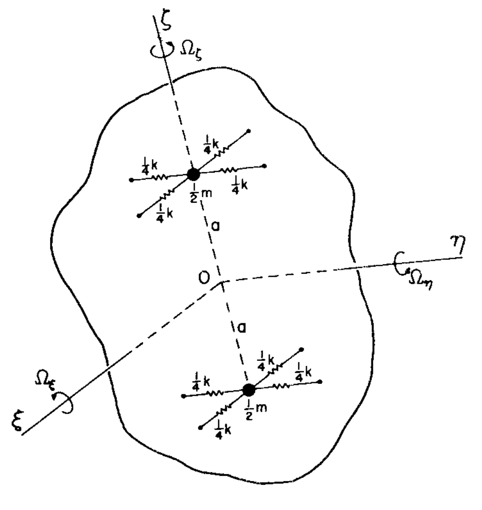

Fig. 1 Torque-free spinning body.

Special Types of Gyroscopic Systems

Let us assume that a rotating structure subjected to conservative torques has been discretized by any of the commonly used schemes (see, for example, Refs. 9 and 10). Then, considering a rigid frame rotating with the structure, where the frame can be identified as a system of axes attached to the structure in its dynamic equilibrium configuration, and measuring the elastic displacements relative to that frame, Lagrange's equations of motion can be written in the matrix form

$$m\ddot{\mathbf{q}}(t) + g\dot{\mathbf{q}}(t) + k\mathbf{q}(t) = \mathbf{0} \tag{40}$$

in which m and k are $n \times n$ real symmetric matrices, g is an $n \times n$ real skew symmetric matrix, and $\mathbf{q}(t)$ is an n-dimensional real vector describing the system configuration, where the elements of $\mathbf{q}(t)$ consist of the angular displacements of the rigid frame and the elastic displacements relative to the frame. Note that the vector $\mathbf{q}(t)$ excludes quantities defining an equilibrium state, such as steady rotations or elastic displacements constant in time, where the latter occur in cases in which the elastic members are deformed in the equilibrium state.

Introducing the $2n$-dimensional state vector

$$\mathbf{x}(t) = \begin{bmatrix} \dot{\mathbf{q}}(t) \\ \mathbf{q}(t) \end{bmatrix} \tag{41}$$

Eq. (40) can be reduced to the matrix form (1) in which I and G have the partitioned forms

$$I = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}, \quad G = \begin{bmatrix} g & k \\ -k & 0 \end{bmatrix} \tag{42}$$

Clearly, I is symmetric and G is skew symmetric. The matrix m *is generally positive definite.* Hence, I *is positive definite if* k *is positive definite.*

Introducing Eqs. (42) into Eq. (15), and recognizing that the multiplication of partitioned matrices and the inversion of block-diagonal matrices can be carried out as if the submatrices were single elements (provided the relative position of the elements is preserved), we can write

$$K = \begin{bmatrix} g^T & -k \\ k & 0 \end{bmatrix} \begin{bmatrix} m^{-1} & 0 \\ 0 & k^{-1} \end{bmatrix} \begin{bmatrix} g & k \\ -k & 0 \end{bmatrix} = \begin{bmatrix} g^T m^{-1} g + k & g^T m^{-1} k \\ k m^{-1} g & k m^{-1} k \end{bmatrix} \tag{43}$$

Moreover, using Eq. (21), we obtain

$$K' = \left[\begin{array}{c:c} m^{-1/2} & 0 \\ \hdashline 0 & k^{-1/2} \end{array}\right] \left[\begin{array}{c:c} g^T m^{-1} g + k & g^T m^{-1} k \\ \hdashline k m^{-1} g & k m^{-1} k \end{array}\right] \left[\begin{array}{c:c} m^{-1/2} & 0 \\ \hdashline 0 & k^{-1/2} \end{array}\right]$$
$$= \left[\begin{array}{c:c} m^{-1/2} g^T m^{-1} g m^{-1/2} + m^{-1/2} k m^{-1/2} & m^{-1/2} g^T m^{-1} k^{1/2} \\ \hdashline k^{1/2} m^{-1} g m^{-1/2} & k^{1/2} m^{-1} k^{1/2} \end{array}\right] \tag{44}$$

As pointed out earlier, there is a large variety of computational algorithms capable of solving efficiently the eigenvalue problem defined by either the real symmetric matrices I and K or by the real symmetric matrix K'.

If, on the other hand, a force-free structure possessing axial symmetry spins initially about the symmetry axis with very high angular velocity, then after some small perturbation the angular velocity about the spin axis remains constant but small nutational motion and elastic motion ensue.[3] In this case, the linearized equations of motion, written in terms of quasi-coordinates, can also be reduced to the form (1). This can be verified in the following example, which solves the problem of Ref. 3 by the new approach.

Illustrative Example

The system shown in Fig. 1 consists of a torque-free symmetric rigid body with its symmetry axis coinciding with axis ζ. The mass moments of inertia of the rigid body alone about axes ξ, η, and ζ are A, $B = A$, and C, respectively. Two equal masses $\frac{1}{2}m$ lie at distances $\zeta = \pm a$ from the fixed center 0, and they are supported by springs as shown in the figure. The body spins originally about axis ζ with large constant angular velocity Ω_ζ, with the masses at rest along the ζ axis. Because of some perturbation, the body acquires the angular velocities Ω_ξ and Ω_η, where $\Omega_\xi \ll \Omega_\zeta$ and $\Omega_\eta \ll \Omega_\zeta$. Moreover, the two masses undergo small elastic displacements, where the displacements are assumed to be antisymmetric, so that the mass at $\zeta = a$ undergoes the displacements u_ξ and u_η and the mass at $\zeta = -a$ undergoes the displacements $-u_\xi$ and $-u_\eta$.

Under the assumptions listed previously, it can be shown that[3]

$$\Omega_\zeta = \Omega = \text{const} \tag{45}$$

Moreover, introducing the new variables

$$w_\xi = \dot{u}_\xi - \Omega u_\eta, \quad w_\eta = \dot{u}_\eta + \Omega u_\xi \tag{46}$$

the equations of motion can be written in the form of the set of first-order differential equations

$$\begin{aligned}
m\omega^2(\dot{u}_\xi - \Omega u_\eta - w_\xi) &= 0 \\
m\omega^2(\dot{u}_\eta + \Omega u_\xi - w_\eta) &= 0 \\
m\dot{w}_\xi + ma\dot{\Omega}_\eta + m\omega^2 u_\xi - m\Omega w_\eta + ma\Omega\Omega_\xi &= 0 \\
m\dot{w}_\eta - ma\dot{\Omega}_\xi + m\omega^2 u_\eta + m\Omega w_\xi + ma\Omega\Omega_\eta &= 0 \\
-ma\dot{w}_\eta + A'\dot{\Omega}_\xi - ma\Omega w_\xi + (C - A')\Omega\Omega_\eta &= 0 \\
ma\dot{w}_\xi + A'\dot{\Omega}_\eta - ma\Omega w_\eta - (C - A')\Omega\Omega_\xi &= 0
\end{aligned} \tag{47}$$

where $A' = A + ma^2$ is the mass moment of inertia of the entire body about a transverse axis and $\omega^2 = k/m$. Equations (47) can be written in matrix form by defining the matrices

$$I = \begin{bmatrix} m\omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & m\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & ma \\ 0 & 0 & 0 & m & -ma & 0 \\ 0 & 0 & 0 & -ma & A' & 0 \\ 0 & 0 & ma & 0 & 0 & A' \end{bmatrix} \tag{48a}$$

$$G = \begin{bmatrix} 0 & -m\omega^2\Omega & -m\omega^2 & 0 & 0 & 0 \\ m\omega^2\Omega & 0 & 0 & -m\omega^2 & 0 & 0 \\ m\omega^2 & 0 & 0 & -m\Omega & ma\Omega & 0 \\ 0 & m\omega^2 & m\Omega & 0 & 0 & ma\Omega \\ 0 & 0 & -ma\Omega & 0 & 0 & (C-A')\Omega \\ 0 & 0 & 0 & -ma\Omega & -(C-A')\Omega & 0 \end{bmatrix} \tag{48b}$$

and the state vector

$$\mathbf{x} = [u_\xi\ u_\eta\ w_\xi\ w_\eta\ \Omega_\xi\ \Omega_\eta]^T \tag{49}$$

It is easy to verify that I is symmetric and positive definite whereas G is skew symmetric. Moreover, we confine ourselves to the case in which G is nonsingular. Hence, the equations of motion (47) are of the form (1), so that the theory presented in this paper is applicable. Before the solution of the eigenvalue problem is carried out, however, it appears desirable to assign numerical values to the system parameters.

Considering the following parameters

$$\begin{gathered} \Omega = 50 \text{ rad/sec}, \quad \omega = 60 \text{ rad/sec} \\ A'/A = 1.1, \quad C/A = 1.5, \quad a = 2 \text{ ft} \end{gathered} \tag{50}$$

the matrices I and G become

$$I = A \begin{bmatrix} 90 & 0 & 0 & 0 & 0 & 0 \\ 0 & 90 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.025 & 0 & 0 & 0.05 \\ 0 & 0 & 0 & 0.025 & -0.05 & 0 \\ 0 & 0 & 0 & -0.05 & 1.1 & 0 \\ 0 & 0 & 0.05 & 0 & 0 & 1.1 \end{bmatrix} \tag{51}$$

and

$$G = A \begin{bmatrix} 0 & -4{,}500 & -90 & 0 & 0 & 0 \\ 4{,}500 & 0 & 0 & -90 & 0 & 0 \\ 90 & 0 & 0 & -1.25 & 2.50 & 0 \\ 0 & 90 & 1.25 & 0 & 0 & 2.50 \\ 0 & 0 & -2.50 & 0 & 0 & 20 \\ 0 & 0 & 0 & -2.50 & -20 & 0 \end{bmatrix} \tag{52}$$

Inserting Eqs. (51) and (52) into Eq. (15), we can write

$$K = G^T I^{-1} G =$$
$$A \begin{bmatrix} 581{,}400 & 0 & 0 & -9{,}000 & 13{,}500 & 0 \\ 0 & 581{,}400 & 9{,}000 & 0 & 0 & 13{,}500 \\ 0 & 9{,}000 & 152.5 & 0 & 0 & 125 \\ -9{,}000 & 0 & 0 & 152.5 & -125 & 0 \\ 13{,}500 & 0 & 0 & -125 & 875 & 0 \\ 0 & 13{,}500 & 125 & 0 & 0 & 875 \end{bmatrix} \tag{53}$$

To obtain K', we must calculate $I^{-1/2}$. To this end, we first solve the eigenvalue problem associated with I, obtain the matrices P and R, where the latter is a diagonal matrix, and use Eq. (29) to write $I^{-1/2} = PR^{-1/2}P^{-1}$. Upon solving the eigenvalue problem associated with I by the Jacobi method, and using Eq. (21), we can calculate

$$K' = I^{-1/2} K I^{-1/2} =$$

$$\begin{bmatrix} 6{,}460 & 0 & 1{,}395.80618 & 0 & 0 & -5{,}854.63279 \\ 0 & 6{,}460 & 0 & 1{,}395.80618 & 5{,}854.63279 & 0 \\ 1{,}395.80618 & 0 & 802.87442 & 0 & 0 & -575.55009 \\ 0 & 1{,}395.80618 & 0 & 802.87442 & 575.55009 & 0 \\ 0 & 5{,}854.63279 & 0 & 575.55009 & 6{,}282.12558 & 0 \\ -5{,}854.63279 & 0 & -575.55009 & 0 & 0 & 6{,}282.12558 \end{bmatrix} \tag{54}$$

The solution of the eigenvalue problem associated with K' yields the eigenvalues ω_r^2 and the modified eigenvectors $\mathbf{y}_r'$ and $\mathbf{z}_r'$ ($r = 1$–3), where the latter are converted into the actual eigenvectors $\mathbf{y}_r$ and $\mathbf{z}_r$ by means of Eqs. (24). The final results, also obtained by the Jacobi method, are

$$\omega_1 = (6.36892174)^{1/2} = 2.5237 \text{ rad/sec} \tag{55a}$$

$$\mathbf{y}_1 = A^{-1/2}\begin{bmatrix} 0 \\ -0.06068106 \\ 3.18717717 \\ 0 \\ 0 \\ 0.48595747 \end{bmatrix}, \quad \mathbf{z}_1 = A^{-1/2}\begin{bmatrix} -0.06068106 \\ 0 \\ 0 \\ -3.18717717 \\ 0.48595747 \\ 0 \end{bmatrix} \tag{55b}$$

$$\omega_2 = (1{,}143.1926)^{1/2} = 33.8111 \text{ rad/sec} \tag{56a}$$

$$\mathbf{y}_2 = A^{-1/2}\begin{bmatrix} 0.04250675 \\ 0 \\ 0 \\ 3.56253837 \\ 0.86835775 \\ 0 \end{bmatrix}, \quad \mathbf{z}_2 = A^{-1/2}\begin{bmatrix} 0 \\ -0.04250675 \\ 3.56253837 \\ 0 \\ 0 \\ -0.86835775 \end{bmatrix} \tag{56b}$$

$$\omega_3 = (12{,}395.4385)^{1/2} = 111.3348 \text{ rad/sec} \tag{57a}$$

$$\mathbf{y}_3 = A^{-1/2}\begin{bmatrix} 0.07498087 \\ 0 \\ 0 \\ -4.59893700 \\ -0.09899568 \\ 0 \end{bmatrix}, \quad \mathbf{z}_3 = A^{-1/2}\begin{bmatrix} 0 \\ 0.07498087 \\ 4.59893700 \\ 0 \\ 0 \\ -0.09899568 \end{bmatrix} \tag{57b}$$

where the eigenvectors $\mathbf{y}_r$, $\mathbf{z}_r$ ($r = 1$–3) are normalized according to the scheme (36). It should be pointed out that the vectors are not entirely independent, as the sign of their elements must be consistent with Eqs. (12).

Note that the same problem was treated in Ref. 3, where the eigenvalues were obtained by solving the characteristic equation numerically. The only exception is that the system of Ref. 3 was damped, whereas the one treated here is not. Hence, the natural frequencies obtained here should compare with the imaginary parts of the eigenvalues of Ref. 3, bearing in mind that the imaginary parts of Ref. 3 represent damped frequencies. The comparison is indeed very satisfactory.

Conclusions

A new method of solution of the eigenvalue problem for gyroscopic systems defined by two real nonsingular matrices, one symmetric and the other skew symmetric, is presented. The method takes full advantage of the fact that one matrix is symmetric and the other is skew symmetric to reduce the eigenvalue problem to a standard form, i.e., one defined by two real symmetric matrices, where the latter resembles in structure the eigenvalue problem of a nonrotating system. Whereas the solution of the original eigenvalue problem must be effected in terms of complex eigenvalues and eigenvectors, the solution of the eigenvalue problem in standard form is in terms of real eigenvalues and eigenvectors. Of course, the two solutions are related. The solution of the eigenvalue problem in standard form can be obtained by a large variety of existing computational algorithms.

It should be pointed out that the present method yields eigenvalues and eigenvectors for the complete structure, i.e., including the rotational motion and the elastic motion. The orthogonality relations and the expansion theorem developed provide the foundation for a true modal analysis, thus enhancing not only the mathematical analysis but also the understanding of the behavior of rotating structures. As an application of the theory developed, the natural frequencies and natural modes of the complete system of Ref. 3 have been calculated. This represents the first time that a complete solution of the eigenvalue problem for a flexible gyroscopic system has been obtained in a systematic manner.

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